MATHEMATICS SPECIALIST

Calculator-free

UNIT 1 Semester 1 2023

Marking Key

Marking keys outline the expectations of examination responses. They help to ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section One: Calculator Free **(51 marks)**

Question 1 (5 marks)

49 is called ‘prime-looking’ because it is composite but not divisible by 2, 3 or 5.

49 is the first prime-looking number.

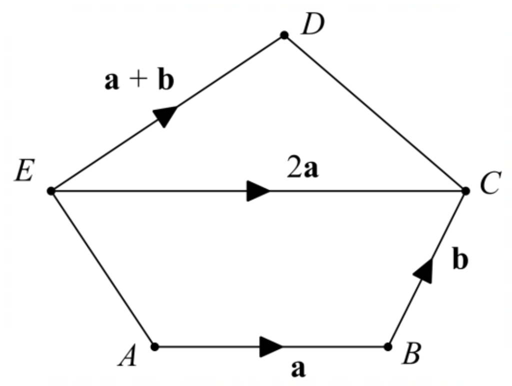
There are 25 prime numbers between 1 and 100 inclusive.

Use include-exclusion principle to show that there are only 3 **prime-looking** numbers between 1 and 100 inclusive.

Solution	Specific behaviours	Point
$\text{Number divisible by } 2 = \frac{100}{2} = 50$ $\text{Number divisible by } 3 = \left\lfloor\frac{100}{3}\right\rfloor = 33$ $\text{Number divisible by } 5 = \frac{100}{5} = 20$ $\text{Divisible by 2 and } 3 = \left\lfloor\frac{100}{6}\right\rfloor = 16$ $\text{Divisible by 3 and } 5 = \left\lfloor\frac{100}{15}\right\rfloor = 6$ $\text{Divisible by 2 and } 5 = \frac{100}{10} = 10$ $\text{Divisible by } 2, 3 \text{ and } 5 = \left\lfloor\frac{100}{30}\right\rfloor = 3$ $\text{Divisible by } 2, 3 \text{ or } 5$ $= 50 + 33 + 20 - 16 - 6 - 10 + 3 = 74$ Prime looking $= 100 - 74 - 25 + 3 - 1$ $= 3$	✓ Determines number divisible by 2, 3 and 5. ✓ Determines number divisible by 2 and 3, 3 and 5 and 2 and 5. ✓ Uses inclusion-exclusion principle to determines number divisible by 2, 3 or 5. ✓ Subtracts 74 and number of primes (25) from 100. ✓ Adds 3 (2, 3, 5 counted twice in primes and divisibility by 2, 3, 5) and subtracts 1 (1 is not prime-looking). (Students using a Venn diagram, should be awarded 2 marks for the entries in the Venn diagram, and the last three marks as above)	1.2.5

Question 2 **(6 marks)**

$ABCDE$ is a pentagon, with $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{EC} = 2\mathbf{a}$ and $\overrightarrow{ED} = \mathbf{a} + \mathbf{b}$.



(a) Determine, in terms of $\mathbf{a}$ and $\mathbf{b}$,

(i) an expression for $\overrightarrow{CD}$. (2 marks)

Solution	Specific behaviours	Point
$\overrightarrow{CD} = \overrightarrow{CE} + \overrightarrow{ED}$ $\overrightarrow{CD} = -2\mathbf{a} + \mathbf{a} + \mathbf{b}$ $\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$	✓ Recognises $\overrightarrow{CE} = -\overrightarrow{EC} = -2\mathbf{a}$ and uses triangle rule. ✓ Determines expression for $\overrightarrow{CD}$.	1.3.4

(ii) an expression for $\overrightarrow{AE}$. (2 marks)

Solution	Specific behaviours	Point
$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CE}$ $\overrightarrow{AE} = \mathbf{a} + \mathbf{b} - 2\mathbf{a}$ $\overrightarrow{AE} = \mathbf{b} - \mathbf{a}$	✓ States vector sum to determine $\overrightarrow{AE}$. ✓ Determines expression for $\overrightarrow{AE}$.	1.3.4

(b) Hence, state two geometric facts about the sides CD and AE. (2 marks)

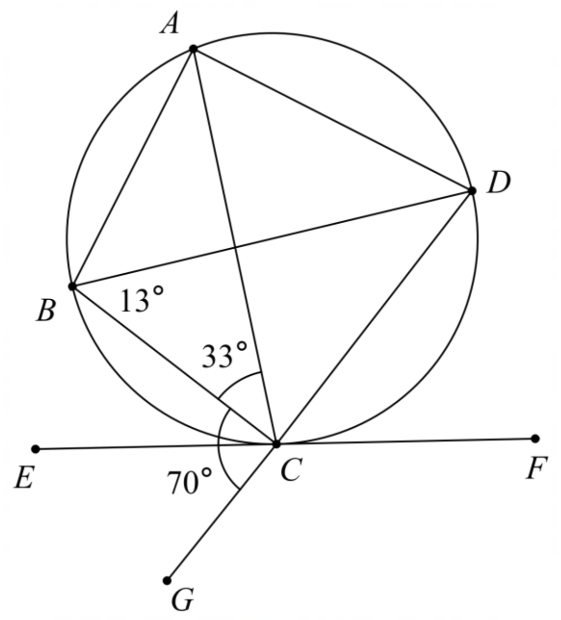
Solution	Specific behaviours	Point
$\overrightarrow{CD}$ and $\overrightarrow{AE}$ are parallel and of equal length	✓ States the vectors are parallel. ✓ States they have the same length.	1.3.2 1.3.12

Question 3 **(8 marks)**

$ABCD$ is a cyclic quadrilateral.

EF is a tangent at C, and DCG is a straight line.

$\angle BCA = 33°$
$\angle GCB = 70°$
$\angle DBC = 13°$



(a) Determine the following angles, giving reasons

(i) $\angle BAD$ (2 marks)

Solution	Specific behaviours	Point
$\angle BCD = 110°$ (angles on straight line) $\angle BAD = 70°$ (opposite angles in a cyclic quadrilateral are supplementary)	✓ Determines $\angle BAD$. ✓ Gives appropriate reason(s).	1.1.9

(ii) $\angle BDA$ (2 marks)

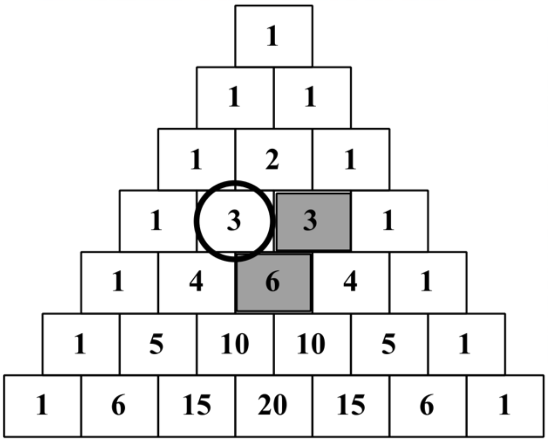
Solution	Specific behaviours	Point
$\angle BDA = 33°$ (angles in same arc)	✓ Determines $\angle BDA$. ✓ Gives an appropriate reason.	1.1.8

(b) Prove that AC passes through the centre of the circle, justifying your answer. (4 marks)

Solution	Specific behaviours	Point
$\angle ACD = 180 - 33 - 70$ $= 77°$ (angles on straight line) $\angle BAD = 77°$ (angle in same arc) $\angle ABC = \angle BAD + 13° = 90°$ $\angle ADC = 90°$ (opposite angles in a cyclic quadrilateral are supplementary) As $\angle ADC = 90°$ then ADC is a semicircle, and hence AC is the diameter.	✓ Determines $\angle ACD$. ✓ Determines $\angle BAD$. ✓ Determines $\angle ADC$. ✓ Explains why AC is the diameter.	1.1.6

Question 4 (6 marks)

The first seven rows of Pascal's triangle are given below.



In this example, the square of the circled number, is equal to sum of the two shaded numbers i.e. $\mathbf{3^2 = 3 + 6}$.

(a) Write down the two numbers in Pascal's triangle that have a sum of 49. (2 marks)

Solution	Specific behaviours	Point
$7^2 = 21 + 28$	✓ Continues Pascal's triangle, and determines at least two of the numbers in the sum. ✓ Determines all numbers and shows the sum.	1.2.9

(b) Using combinations prove this pattern holds for any second element of any row. (4 marks)

Solution	Specific behaviours	Point
$\binom{n}{2} + \binom{n+1}{2} = \frac{n!}{2!\,(n-2)!} + \frac{(n+1)!}{2!\,(n-1)!}$ $= \frac{n(n-1)}{2!} + \frac{n(n+1)}{2!}$ $= \frac{n^2 - n + n^2 + n}{2!}$ $= n^2$	✓ Correctly writes the two terms of the sum as combinations. ✓ Substitutes into combination formula. ✓ Expands factorials and simplifies. ✓ Expands brackets and arrives at n^2.	1.2.9

Question 5 **(9 marks)**

(a) Given ${}^{n}P_{r} = 90$ and ${}^{n}C_{r} = 45$, determine the value of n and the value of r. (4 marks)

Solution	Specific behaviours	Point
$90 = \frac{n!}{(n-r)!} \quad (1)$ $45 = \frac{n!}{r!\,(n-r)!} \quad (2)$ $(1) \div (2) \Rightarrow \frac{90}{45} = r!$ $r! = 2 \Rightarrow r = 2$ Subtitute $r = 2$ into (1) $90 = \frac{n \times (n-1) \times (n-2)!}{(n-2)!}$ $n^2 - n - 90 = 0$ $(n+9)(n-10) = 0$ $\therefore n = 10$	✓ Sets up two equations using permutation and combination formulae. ✓ Solves for r. ✓ Substitutes $r = 2$ into either (1) or (2). ✓ Solves for n.	1.2.3 1.2.8

(b) (i) Show that the number of different 9-letter arrangements of the word 'TENNESSEE' is $\frac{3}{4} \times 7!$. (2 marks)

Solution	Specific behaviours	Point
Arrangements of TENNESSEE: $\frac{9!}{4! \times 2! \times 2!} = \frac{9 \times 8 \times 7!}{4 \times 3 \times 2 \times 2}$ $= \frac{3}{4} \times 7!$	✓ Determines arrangements of Tennessee using factorials. ✓ Correctly simplifies to required answer.	1.2.3 1.2.4

(ii) Lara believes that the number of different 11-letter arrangements of the word 'MISSISSIPPI' is greater than of the arrangements of the word 'TENNESSEE'. Is she correct? Justify your answer mathematically. (3 marks)

Solution	Specific behaviours	Point
Arrangements of MISSISSIPPI: $\frac{11!}{4! \times 4! \times 2!}$ $= \frac{11 \times 10 \times 9 \times 8 \times 7!}{4 \times 3 \times 2 \times 4 \times 3 \times 2 \times 2}$ $= \frac{11 \times 10}{4 \times 4} \times 7!$ As $\frac{110}{16} > \frac{3}{4}$, then Lara is correct	✓ Determines arrangements of Mississippi using factorials. ✓ Works towards simplifying factorial expression. ✓ Concludes Lara is correct with a valid justification.	1.2.3 1.2.4

Question 6 (9 marks)

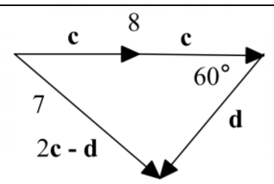
(a) A parallelogram $ABCD$ is defined by vectors $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.

If $\mathbf{a} = (x-1)\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + (x^2-1)\mathbf{j}$, determine the value(s) of x, such that $ABCD$ is a rectangle. (3 marks)

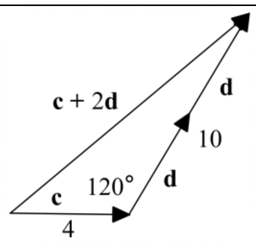
Solution	Specific behaviours	Point
$\mathbf{a} \bullet \mathbf{b} = 0$ $2(x-1) + (x^2-1) = 0$ $x^2 + 2x - 3 = 0$ $(x+3)(x-1) = 0$ $x = -3, 1$	✓ Indicates that $\mathbf{a} \bullet \mathbf{b} = 0$. ✓ Determines dot product ✓ Solves for x.	1.3.11 1.3.13

(b) The angle between two vectors, $\mathbf{c}$ and $\mathbf{d}$ is $60°$, with $|\mathbf{c}| = 4$, $|2\mathbf{c} - \mathbf{d}| = 7$ and $|\mathbf{c}| < |\mathbf{d}|$.

(i) Show, with aid of a diagram, that $|\mathbf{d}| = 5$. (3 marks)

Solution	Specific behaviours	Point
 $7^2 = 8^2 + |\mathbf{d}|^2 - 2 \times 8|\mathbf{d}| \cos 60°$ $49 = 64 + |\mathbf{d}|^2 - 8|\mathbf{d}|$ $|\mathbf{d}|^2 - 8|\mathbf{d}| + 15 = 0$ $(|\mathbf{d}| - 3)(|\mathbf{d}| - 5) = 0$ $|\mathbf{d}| = 5$ as $|\mathbf{c}| < |\mathbf{d}|$	✓ Draws diagram including angle, and relevant side lengths. ✓ Substitutes into cosine rule. ✓ Rearranges and factorises. ✓ Solves for $|\mathbf{d}|$.	1.3.2 1.3.3 1.3.4

(ii) Hence, or otherwise, determine the length of $\mathbf{c} + 2\mathbf{d}$. (3 marks)

Solution	Specific behaviours	Point
 $|\mathbf{c} + 2\mathbf{d}|$ $= \sqrt{4^2 + 10^2 - 2 \times 4 \times 10 \cos 120°}$ $|\mathbf{c} + 2\mathbf{d}| = \sqrt{116 + 40}$ $|\mathbf{c} + 2\mathbf{d}| = \sqrt{156} = 2\sqrt{39}$	✓ Draws a diagram, or indicates in working that required angle is 120°. ✓ Substitutes into cosine rule. ✓ Determines length of $\mathbf{c} + 2\mathbf{d}$.	1.3.2 1.3.3 1.3.4

Question 7 **(8 marks)**

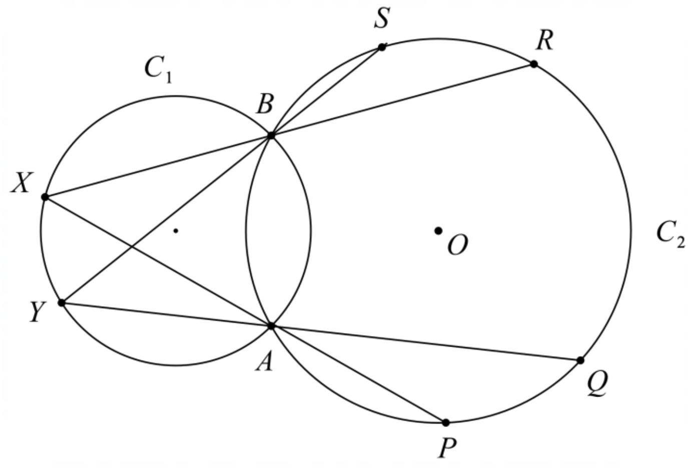
Two circles, C_1 and C_2, intersect at A and B.

Two points X and Y are on C_1.

The line XA extended intersects with C_2 at P, and YB extended intersects with C_2 at S.

The line XB extended intersects with C_2 at R, and YA extended intersects with C_2 at Q.

O is the centre of C_2.



(a) Prove that $\angle PAQ = \angle SBR$, giving reasons. (3 marks)

Solution	Specific behaviours	Point
$\angle YAX = \angle YBX$ (angle in the same arc) $\angle PAQ = \angle YAX$ and $\angle SBR = \angle YBX$ (vertically opposite angles) $\therefore \angle PAQ = \angle SBR$	✓ States that $\angle YAX = \angle YBX$ with valid reason. ✓ Uses vertically opposite angles to show that $\angle PAQ = \angle YAX$ and $\angle SBR = \angle YBX$. ✓ Concludes that $\angle PAQ = \angle SBR$.	1.1.8

(b) Prove the chords PR and QS are congruent. (5 marks)

Solution	Specific behaviours	Point
Let $\angle PAQ = \angle SBR = \alpha$ $\angle POQ = 2\alpha$ and $\angle SOR = 2\alpha$ (angles at centre are twice angles at circumfrence) $\angle POR = \angle POQ + \angle QOR$ $\angle POR = 2\alpha + \angle QOR$ Chord PR subtends $\angle POR$ $= 2\alpha + \angle QOR$ $\angle QOS = \angle SOR + \angle QOR$ $\angle QOS = 2\alpha + \angle QOR$ Chord QS subtends $\angle QOS$ $= 2\alpha + \angle QOR$ $PR \equiv QS$ as chords subtending equal angles at the centre of a circle have equal length	✓ Uses angle at the centre is twice the angle at the circumference to determine $\angle POQ$ and $\angle SOR$. ✓ Determines an expression for $\angle POR$. ✓ Determines an expression for $\angle QOS$. ✓ States that the chords subtend the angle $2\alpha + \angle QOR$ at the centre. ✓ Explains why PR and QS are congruent.	1.1.7 1.1.10

End of Calculator Free Section